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LETTER TO THE EDITOR

Temperley–Lieb words as valence-bond ground states

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Abstract. Based on the Temperley–Lieb algebra we define a class of one-dimensional Hamiltonians with nearest- and next-nearest-neighbour interactions. Using the regular representation we present ground states of this model as words of the algebra. Two-point correlation functions can be computed employing the Temperley–Lieb relations. Choosing a spin-$\frac{1}{2}$ representation of the algebra we obtain a generalization of the (q-deformed) Majumdar–Ghosh model. The ground states become valence-bond states.

In this letter we present a class of one-dimensional Hamiltonians $H = H(a, b, q)$ with nearest- and next-nearest-neighbour interaction. The Hamiltonian is given in terms of elements of a Temperley–Lieb algebra [1] and has the structure of the Majumdar–Ghosh model [2–6]. With specific representations of this algebra one obtains various quantum spin chains. The functions $a = a(q)$ and $b = b(q)$ determine the next-nearest-neighbour interaction and can be chosen such that the ground state can be given explicitly. In a graphical form of the regular representation of the Temperley–Lieb algebra [7, 8] these ground states have a particular simple form. They are related to valence-bond spin states. We also calculate correlation functions using the graphical representation.

A Temperley–Lieb algebra $T_N(q)$ is defined by the following relations on the generators e_i, $i = 1, 2, \ldots, N-1$:

$$e_i e_i = x e_i = (q + q^{-1}) e_i \tag{1a}$$

$$e_i e_{i\pm1} e_i = e_i \tag{1b}$$

$$e_i e_j = e_j e_i \qquad (j \neq i \pm 1). \tag{1c}$$

Here we consider the case where q is real. These algebras appear as centralizer algebras of the quantum group $U_qSU(2)$ [7, 9]. Therefore, $U_qSU(2)$ invariant models naturally show an underlying Temperley–Lieb structure. However, our results can be used for models having other quantum group symmetries as well [10–13].

We define the Hamilton operator as an abstract element of the Temperley–Lieb algebra. We use the two-point correlation operators defined in [8]. These operators preserve the quantum group symmetry of the respective representations of the Temperley–Lieb algebra. Two types of two-point operators can be defined by the recursive relations

$$\begin{aligned} g_{l,l+1} &= g^{\pm}_{l,l+1} = e_l - (q + q^{-1})^{-1} && 1 \leqslant l \leqslant N-1 \\ g^{\pm}_{l,m} &= -q^{\pm1} g^{\pm}_{l,n} g^{\pm}_{n,m} - q^{\mp1} g^{\pm}_{n,m} g^{\pm}_{l,n} && 1 \leqslant l < n < m \leqslant N \\ g^{\pm}_{m,l} &= q^{\mp4} g^{\pm}_{l,m} && 1 \leqslant l < m \leqslant N. \end{aligned} \tag{2}$$

In the following we use only $g^{\pm}_{l,m}$ with $l < m$. Note that the definition is in terms of generators e_l of $T_N(q)$, independent of their realization.

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We employ both $g^+_{l,m}$ and $g^-_{l,m}$ to define the Hamilton operator

$$H^{(N)}(a,b,q) = \frac{2-x^2}{2x(x^2-1)} \sum_{i=1}^{N-2} h_{i,i+1,i+2} + 2a(q)(N-2)(2-x^2)^2 \tag{3}$$

with

$$\begin{aligned} h_{i,i+1,i+2} = {} & 2g_{i,i+1} + 2((x^2-2)a(q)-1)g_{i+1,i+2} + a(q)(g^+_{i,i+2} + g^-_{i,i+2}) \\ & + b(q)(g^+_{i,i+2} - g^-_{i,i+2}). \end{aligned} \tag{4}$$

The operator $h_{i,i+1,i+2}$ acts on sites i, $i+1$, $i+2$ and involves the generators e_i and e_{i+1}. The value of x is given by the Temperley–Lieb relation (1a). The arbitrary functions $a = a(q)$ and $b = b(q)$ are weighting factors of the symmetric and antisymmetric contribution of $g^+_{i,i+2}$ and $g^-_{i,i+2}$ to the Hamiltonian. For $q = 1$ the Hamiltonian is independent of b since $g^+_{l,m} = g^-_{l,m}$. It is useful to notice how the Hamiltonian changes if we replace q by $1/q$. We have the identity

$$H^{(N)}(a,b,q) = H^{(N)}(a,-b,1/q). \tag{5}$$

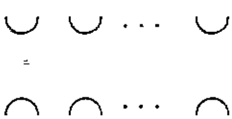
The coefficients in (4) are chosen such that simple ground states can be found. To achieve this we use the regular representation of $T_N(q)$ on boundary diagrams [7]. We find the ground states to be specific Temperley–Lieb words. A word in this context means a straight product of generators and is realized by a single boundary diagram.

A boundary diagram is given by two rows of N upper and N lower points with N non-intersecting lines connecting the points such that any point is connected to a single other point. A generator is realized as

$e_i =$  (6)

The composition of two words is defined as stacking the corresponding diagrams on top of each other and identifying the lower points of the first with the upper points of the second diagram. Any closed line appearing in this process is discarded from the diagram and replaced by a factor $x = q + q^{-1}$. This reflects relation (1a). The other Temperley–Lieb relations (1b) and (1c) can easily be verified by drawing the corresponding diagrams. By successive composition of generators (6) all distinct words of $T_N(q)$ are realized as different boundary diagrams. Thus, for calculations in the regular representation of $T_N(q)$ we can take all possible boundary diagrams as a basis of the representation space.

In the following we give eigenvectors and eigenvalues of $H^{(N)}$. First, we take N even. Employing boundary diagrams it is straightforward to show that (3) has the eigenvector

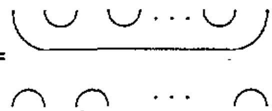
$v_1 = e_1 e_3 \ldots e_{N-1} =$  . (7)

The additive constant in (3) is chosen such that its eigenvalue is zero.

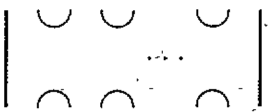
If we impose a condition on the functions $a(q)$ and $b(q)$:

$$a = (2 - x(q - q^{-1})b)/(x^2 - 1) \tag{8}$$

we find two further eigenwords with eigenvalue zero:

$v_2 = (e_2 e_4 \ldots e_{N-2}) v_1 =$  (9)

and

$v_3 = e_2 e_4 \ldots e_{N-2} =$  . (10)

For an odd number of sites N and condition (8) we have two eigenwords that are given by diagram (10) without the left or right vertical line. We denote these words as $v_4 = e_1 e_3 \ldots e_{N-2}$ and $v_5 = e_2 e_4 \ldots e_{N-1}$ respectively. Their eigenvalues can be computed as $\mp b/2\,(q^2+q^{-2})(q-q^{-1})(q+q^{-1})^{-1}$ for v_4 and v_5 respectively.

For the following we impose condition (8). We investigate whether we can choose the function $b(q)$ such that the eigenstates v_i given above become ground states of the Hamiltonian $H^{(N)}$. We can show that $H^{(N)}$ has a ground state v_i if the function $b(q)$ is bounded by

$$b \leqslant (\geqslant) \frac{q^3+2q+2q^{-1}+q^{-3}}{q^5+2q^3-2q^{-3}-q^{-5}} \qquad \text{for } |q|>1 (|q|<1). \tag{11}$$

The relation between the cases $|q|>1$ and $|q|<1$ reflects the symmetry (5).

For a proof, first consider the case N even. One can think of $H^{(N)}$ as the sum of $N/2-1$ Hamilton operators $H^{(4)}$ each involving only three generators. This can be seen by grouping together $h_{i,i+1,i+2}$ and $h_{i+1,i+2,i+3}$ (i odd) in definition (3). The operator $H^{(4)}$ can be diagonalized (see below). Hence, the lowest eigenvalue of $H^{(N)}$ is bounded from below by $N/2-1$ times the lowest eigenvalue of $H^{(4)}$. We take the 14 possible boundary diagrams with $N=4$ as a basis to write $H^{(4)}$ in the regular representation. Diagonalization of this 14×14 matrix gives the eigenvalues

$$\begin{aligned}
\lambda_1 &= 0\\
\lambda_2 &= 2-(q-q^{-1})b(q)\\
\lambda_3 &= 1-(q-q^{-1})(q+q^{-1})^{-1}b(q)\\
\lambda_4 &= 1-(q^3-q^{-3})(q+q^{-1})^{-1}b(q)\\
\lambda_5 &= 1-(q^5+2q^3-2q^{-3}-q^{-5})(q^3+2q+2q^{-1}+q^{-3})^{-1}b(q)\\
\lambda_{6/7} &= 1-\frac{1}{2}(q^3+2q+2q^{-1}+q^{-3})^{-1}[(q^5+2q^3+q-q^{-1}-2q^{-3}-q^{-5})b(q)\\
&\qquad \pm[(q^{10}+6q^6-2q^4+q^2-12+q^{-2}-2q^{-4}+6q^{-6}+q^{-10})b^2(q)\\
&\qquad -8(q^4+q^2-q^{-2}-q^{-4})b(q)+4(q+q^{-1})^2]^{\frac{1}{2}}].
\end{aligned}$$

The degeneracy of the first eigenvalue is seven and of the second two. The others are non-degenerate. From this result we can conclude that, if we have equation (11), all eigenvalues of $H^{(4)}$ are greater or equal to zero. Hence, for functions $b(q)$ that fulfil conditions (8) and (11), v_1, v_2 and v_3 are ground states of the Hamiltonian $H^{(N)}$ because their eigenvalue is zero.

For N odd we can analogously view $H^{(N)}$ as the sum of $(N-3)/2$ operators $H^{(4)}$ and one operator $H^{(3)}$. Making use of the five different boundary diagrams for $N=3$ we find the eigenvalues of $H^{(3)}$ are

$$\mu_1 = 1 - \frac{b(q)}{2}(q^2 - q^{-2})$$
$$\mu_{2/3} = \pm\frac{b(q)}{2}(q^2 + q^{-2})(q - q^{-1})(q + q^{-1})^{-1}.$$

The eigenvalues μ_2 and μ_3 are twofold degenerate while μ_1 is non-degenerate. If we impose condition (11), μ_1 is always positive. Note that for (8) the values μ_2 and μ_3 are also eigenvalues of $H^{(N)}$, with eigenvector v_4 and v_5 respectively. Thus, imposing both conditions (8) and (11) a ground state of $H^{(N)}$ is given by v_4 or v_5 depending on the values of q and b. Explicitly, for $|q| \geqslant 1$ and $b \leqslant 0$ ($b \geqslant 0$) we have v_4 (v_5) as the ground state of $H^{(N)}$. For $|q| \leqslant 1$ and $b \leqslant 0$ ($b \geqslant 0$) we find the ground state v_5 (v_4).

If we drop condition (8), v_1 remains the eigenstate of $H^{(N)}$ for N even. For this case we wish to remark that one can choose two functions $a(q)$ and $b(q)$ within certain bounds to make v_1 a ground state of the Hamiltonian.

Next, we calculate correlation functions for the ground states v_1 and v_2. Again, the computation is completely general in terms of boundary diagrams.

For a graphical calculation of the correlation functions we have to restrict our attention to the words of the left-sided ideal that is generated by $v_1 = e_1 e_3 \ldots e_{N-1}$ (N even). The diagrams with a lower part as in v_1 and an arbitrary upper part constitute the ideal $\mathcal{S}$. Thus, the ground states v_1 and v_2 belong to $\mathcal{S}$.

Defining a transposed diagram as the up-side-down reflected diagram, the scalar product $\langle s_1|s_2\rangle$ of two elements s_1 and s_2 of $\mathcal{S}$ can be defined as follows [8]. If one removes any *loop from the diagram representing* ${s_1}^{\mathrm{T}} s_2$, *the resulting diagram is always* v_1 *with a factor* depending on s_1 and s_2. The scalar product is given by this factor:

$${s_1}^{\mathrm{T}} s_2 = \langle s_1|s_2\rangle \qquad (12)$$

For example, it is easy to see that

$$\langle v_1|v_1\rangle = \langle v_2|v_2\rangle = x^{N/2}. \qquad (13)$$

Thus, we can calculate the expectation value of the correlation operators $g^{\pm}_{l,n}$ for a state v in $\mathcal{S}$. We evaluate the diagrams

$$v^{\mathrm{T}} g^{\pm}_{l,m} v = \langle v|g_{l,m}|v\rangle \qquad (14)$$

using the recurrence relation (2) expressed in terms of diagrams. By induction on k we can show that the correlation functions for the states v_1 and v_2 for all N are given by

$$\langle v_i|g^{\pm}_{l,l+k}|v_i\rangle = \begin{cases} (x - 1/x)x^{N/2} & \text{for } i = 1,\ k = 1 \text{ and } l \text{ odd} \\ (x - 1/x)x^{N/2} & \text{for } i = 2,\ k = 1 \text{ and } l \text{ even} \\ & \text{and for } i = 2,\ l = 1,\ l + k = N \\ 0 & \text{otherwise.} \end{cases} \qquad (15)$$

Thus, only sites connected by an upper line in $v_{1/2}$ have a non-zero correlation.

Note that we find trivial short-range correlations, although the operators $g^{\pm}_{l,l+k}$ have a non-local structure (2). The ground states do not have any long-range order. This can be expected from the interpretation of the words v_1 and v_2 in terms of spins that we give in the following paragraph for the representation (16) with spin $\frac{1}{2}$.

So far we have used the regular representation of the Temperley–Lieb algebra producing results independent of special representations. We now take the $U_qSU(2)$ spin-$\frac{1}{2}$ representation of $T_N(q)$ that is defined by [8,9]

$$e_l = -\frac{1}{2}\left(\sigma_l^x\sigma_{l+1}^x + \sigma_l^y\sigma_{l+1}^y + \frac{q+q^{-1}}{2}(\sigma_l^z\sigma_{l+1}^z - 1) + \frac{q-q^{-1}}{2}(\sigma_l^z - \sigma_{l+1}^z)\right) \tag{16}$$

where σ^x, σ^y and σ^z are Pauli matrices. For this representation the $g_{l,m}^{\pm}$ (2) are $U_qSU(2)$ invariant generalizations of the $SU(2)$ invariant scalar product [8]

$$g_{l,m}^{q=1} = -\tfrac{1}{2}\sigma_l \cdot \sigma_m. \tag{17}$$

The resulting Hamiltonian $H^{(N)}$ is $U_qSU(2)$ invariant. We wish to know the spin configuration states that correspond to the boundary diagrams v_i, i.e. the ground states of this quantum chain for conditions (8) and (11). To achieve this consider the special case

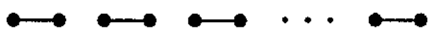
$$b(q) = 0 \qquad \text{and} \qquad a(q) = 2/(x^2 - 2). \tag{18}$$

With the chosen representation and this condition $H^{(N)}$ becomes the Hamiltonian of the q-deformed Majumdar–Ghosh model [2,6]

$$H^{\mathrm{MG}} = \sum_{i=1}^{N-2} P_{i,i+1,i+2}^{3/2} \tag{19}$$

where $P_{i,i+1,i+2}^{3/2}$ is the projector onto the (q-deformed) quartet of the spins at sites i, $i+1$ and $i+2$.

The Majumdar–Ghosh model is known to have a valence-bond ground state [3,4,6]. Denoting the spin-$\frac{1}{2}$ representation space of a site by a dot, and a singlet combination of two adjacent spins by a short line, they can be given pictorially as

$w_1 =$  (20)

$w_2 =$  (21)

for an even number of sites. For N odd the ground state has the form of (21) without the dot to the very left or very right.

One can easily verify that w_1 and w_2 are, in fact, ground states. The action of any $P_{i,i+1,i+2}^{3/2}$ on either of these states gives zero since two of the three spins of sites i, $i+1$ and $i+2$ are in a singlet configuration. Thus, both states are eigenvectors of H^{MG} with zero eigenvalue. Also, all possible eigenvalues of H^{MG} are greater or equal to zero because it is a sum of projectors.

Alternatively, we have the ground states v_1, v_2 and v_3 (respectively v_4 or v_5 for N odd), where v_1 and v_2 belong to the ideal S. For representation (16), S is known to represent the $U_qSU(2)$ scalar states [7,8].

At this point the correspondence between the diagrams v_i and spin configurations w_j becomes clear. A line connecting two upper points in a boundary diagram is represented by the singlet configuration of the spins at the corresponding sites. For this identification one can easily show that the action of the Temperley–Lieb algebra on a diagram of S is mirrored by the action of matrices (16) on the matching spin state. In this way the two words v_1 and v_2 are represented by the two $U_qSU(2)$ ground-state singlets w_1 and w_2 repectively (with the spins of sites 1 and N in the singlet combination). The correct normalization of the spin states can be calculated from (13). Further analysing the action of $T_N(q)$, we find that the word v_3 corresponds to a linear combination of the ground-state triplet and the singlet that are both pictorially described by w_2. Finally, for N odd, v_4 and v_5 correspond to w_2 without the left and right dot respectively.

Recently generalizations of the Majumdar–Ghosh model have been discussed. A class of $SU(2)$ symmetric antiferromagnetic chains with valence-bond ground state can be found in [15]. Takano has introduced a generalization of the projection operators which appear in the formulation of the Majumdar–Ghosh model [16]. A q-deformation of the $SU(2)$ symmetric model has been given in [6]. With representation (16) the class of Hamiltonians $H^{(N)}$ is an extension of the latter $U_qSU(2)$ symmetric Majumdar–Ghosh chain. Note that the valence-bond ground states remain unchanged for a range of functions $a(q)$ and $b(q)$. A similar phenomenon has already been found for a spin-1 chain with matrix-product ground state [17]. We have not investigated whether our Hamiltonian is massless or not [5].

Choosing a different representation we can define further models with the same property. It is possible to represent a Temperley–Lieb algebra on quantum chains with $n = 2s + 1$ states per site. Such a representation of $T_N(q)$ is given through the matrix elements [11]

$$\langle m_i, m_{i+1}|e_i|m'_i, m'_{i+1}\rangle = (-1)^{m_i - m'_i} p^{m_i + m'_i} \delta_{m_i + m_{i+1},0}\delta_{m'_i + m'_{i+1},0} \tag{22}$$

where m_i is the spin variable at site i with $-s \leqslant m_i \leqslant s$. The value of p can be calculated from

$$q + q^{-1} = x = [n]_p \equiv (p^n - p^{-n})(p - p^{-1})^{-1}. \tag{23}$$

For $s = 1/2$ this reduces to the representation given in (16). The resulting Hamiltonians $H^{(N)}$ are $U_pSU(n)$ symmetric [11, 12]. In general the matrices (22) realize the projector onto the $U_pSU(n)$ singlet at two adjacent sites [10, 11] which is realized via the branching rule

$$n \times \bar{n} = (n^2 - 1) + 1. \tag{24}$$

Thus, even for $s > 1/2$ the ground state is of valence-bond type. For example, we can expect the ground state of the $U_pSU(3)$ symmetric model to be tenfold degenerate (one octet and two singlets).

We have given the correspondence between specific words of the Temperley–Lieb algebra and vectors of the spin-$\frac{1}{2}$ configuration space. The general relation of words of $T_N(q)$ and spin states will be discussed in a future publication.

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